CONTROL BANDS FOR TRACKING OPTIMAL PORTFOLIOS IN THE PRESENCE OF CONSTANT AND PROPORTIONAL TRANSACTION COSTS

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Abstract. The vast majority of research related to optimal asset allocation strategies in the presence of transaction costs, requires formulation of highly sophisticated numerical schemes for the estimation of no-transaction bands; moreover, the optimization objectives examined are far less compared to the number of works that assume frictionless trading. In this article, we point out that an investor may alternatively try to track an optimal portfolio as derived in the frictionless case under any optimization objective (HARA utility, minimization of probability of ruin, maximization of probability of reaching a target etc.) by applying a loss function that reflects his/her risk preferences. We focus in the two-asset case (one riskless and one risky) and assume a fixed cost per transaction plus a cost proportional to the change in the risky fraction process. Using a recently proposed transformation of the risky fraction process (Nagai, 2005), we derive optimal allocation policies for the quadratic loss case, using two alternative methods. First, we calculate no transaction bands for investors who choose the boundaries of the bands and the optimal rebalancing actions so that they minimize average cost (opportunity cost / tracking error and transaction cost) per transaction cycle. In the second case, the objective is to minimize the expected discounted squared tracking error plus transaction costs over an infinite horizon. On that purpose, similar to Suzuki and Pliska (2004), we use impulse control theory in a continuous-time, dynamic setting and characterize the optimal strategy in terms of a quasi-variational inequality. For both formulations, we derive explicit solutions, which we will use in a forthcoming version of the article, to perform sensitivity analysis for the control bands with respect to the market parameters and the magnitude of the transaction costs.

Keywords: risky fraction process; stochastic impulse controls; control bands; quasi-variational inequalities.

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1. Introduction

Since R. Merton’s [24] pioneering work on optimal consumption/investment decisions for investors that may place proportions of their wealth in risky assets (“stocks”) whose prices are described by geometric Brownian motions and a bank account (or “bond”) paying a fixed interest rate under the objective of maximizing lifetime HARA (hyperbolic absolute risk aversion) utility of consumption, several articles have emerged in the literature using the same market specifications but different objective functions. To name a few, Pliska [29] derived optimal strategies for investors that aim to maximize exponential utility of terminal wealth, Browne [10], [11] related the probability of achieving a given target performance to the time it takes to achieve it and Young [32] presented strategies that minimize the probability of lifetime ruin. Depending on the optimization objective, the optimal asset allocation may be constant as in Merton or Pliska, or state (wealth) dependent as in Browne or Young; in the risky wealth-risk free wealth space the former result postulates that the portfolio holdings should be located on the so-called Merton line. In the aforementioned models, information arrives continuously and, since investors trade costlessly, optimal policies entail continuous trading; hence following such strategies in the presence of transaction costs will lead to immediate ruin.

Since the early nineties, there has been a wave of research focusing on the removal of one of the most significant simplifying assumptions of Merton’s model: frictionless trading. Efforts for the removal of the frictionless market hypothesis date back to the path-breaking work of Davis and Norman [15] who assumed a cost proportional to the size of each transaction for investors that may invest in a single risky and a riskless asset, aiming to maximize lifetime HARA utility of consumption. Thereafter various articles have appeared studying the optimal transaction policy for an agent facing proportional transaction costs in the financial markets. Shreve and Soner [30] refined the work of Davis and Norman using viscosity solutions, Dumas and Luciano [17] studied the problem of maximizing HARA utility of terminal wealth in the limit as the horizon gets very large, whereas Gennotte and Jung [18] and Liu and Lowenstein [22] focused on CRRA (constant relative risk aversion) utility of terminal wealth. More recently, Nazareth [27] formulated and numerically solved the same problem in which
the constant of proportionality for the transaction costs is random, for investors that maximize HARA utility of lifetime consumption. Demchuk [16] assumed that transaction costs are represented by a concave function of the size of the trade in the risky asset and solved for investors with CRRA utility of terminal wealth. Optimal control actions in markets containing one risky and one riskless asset and rebalancing entails proportional transaction costs, are of a “local time” type, i.e. the fundamental process can move freely inside a prescribed region. If it reaches its boundary, the controller will simply hold it inside the region by performing the minimal action to avoid crossing of the boundary. Hence, policies are totally specified by the boundaries of the “non-intervention” region. The value function of the optimization problem is typically characterized as a solution of a variational inequality.

Morton and Pliska [25], observed that optimal trading strategies derived by models that incorporate transaction costs proportional to the amount traded, are not of finite variation; thus these strategies still consist of making infinitesimally small transactions which is not the case in real world. They assumed transaction costs proportional to wealth and derived control bands for investors aiming to maximize their long-run growth rate of wealth. The optimal policy in their model is of finite variation: each time the bond-stock proportion hits the boundaries of the no-transaction band the investor brings it back to an optimal level within the band. Bielecki and Pliska [7] and Bielecki et al. [8] extended the aforementioned model to risk sensitized growth-rate optimizing criteria. Korn [19], [20], added a fixed cost part to the transaction costs part that is proportional to the amount traded in the risky asset. Using the impulse control method, he obtained optimal strategies that also consist of finitely many actions on finite time intervals; he solved the impulse control problem for an investor who maximizes his/her exponential utility of terminal wealth. In this case, the presence of the fixed cost component forces the controller to move the underlying process away from the boundary of the “non-intervention” region. Here, the value function is characterized as a solution of quasi-variational inequalities (qvi). Later, Oksendal and Sulem [28], based on the theory of viscosity solutions applied to quasi-variational inequalities, presented a numerical scheme that optimizes lifetime HARA utility of consumption. Zakamouline [34], [35] presented numerical schemes for investors aiming to maximize CRRA utility of terminal wealth and Liu [23] derived optimal impulse control bands for investors seeking to optimize
lifetime CARA utility of consumption. The reader should note that practically all research work in transaction costs models is done for simple two-asset markets; due to computational intractability, there is only a handful of articles that consider (correlated) multi-asset markets, see Akian et al. [1], [2] and Atkinson and Mokkhavesa [3]. For some survey articles, the interested reader may also consult Cadenillas [12] and Zariphopoulou [35].

In a recent article, Korn [21] highlighted the difficulties related to the application of transaction cost models in real world tasks. The vast majority of methods require formulation of highly sophisticated numerical schemes for the derivation of optimal allocation rules; thus, it is difficult for a practitioner to derive optimal control bands within which his/her bond-stock proportions should lie and rebalancing points to which the proportions should be driven when they hit the boundaries of the bands. Moreover, the optimization objectives examined are far less compared to the models that adopt the frictionless markets hypothesis; for instance, probability-related objectives as in Browne [10], [11] or Young [32] and Young and Bayraktar [4] have not been examined yet. In this work, we point out that to enhance tractability, an investor may alternatively try to track an optimal portfolio as derived under the frictionless markets hypothesis under any optimization objective, using an appropriate loss function that reflects his/her risk aversion. Based on a transformation of the risky fraction process recently proposed by Nagai [26] and inspired by a simple cash inventory model presented in Carlin and Taylor [14] (section 15.4), we present a simple method for the derivation of optimal rebalancing rules. Furthermore, we use Nagai’s transformation to apply an impulse control model similar to the one presented in Pliska and Suzuki [31] and Cadenillas and Zapatero [13] and compare the results obtained by the two alternative methods. To illustrate our methodology we examine a loss function that penalizes squared deviations from the desired proportions in the original scale.

The plan for our paper is as follows. In section 2, we display our two-asset market model, formulate Nagai’s [26] transformation for the risky fraction process and present a precise statement of the portfolio manager’s optimization objective. In the third section, we explain how optimal trading strategies can be computed via standard diffusion theory for investors
that minimize average cost per transaction cycle. The average cost is comprised by two components:

- a transaction cost that is linear in the change of the (transformed) risky fraction process that occurs in every transaction
- an opportunity cost/tracking error that is dependent on the investor’s risk preferences via an associated loss function.

Similar to the impulse control models presented in Korn [19], [20], Oksendal and Sulem [28], Zakamouline [33] and Liu [23], the optimal strategy is characterized by four unknown parameters $L, l, u, U$. If the proportion of the risky asset hits level $L$ (or $U$), then a transaction is made so that it resumes at level $l$ (or $u$). Estimation of the inner ($l, u$) and outer boundaries ($L, U$) pertains to the solution of a system of four (quite complex) nonlinear equations. Computations can be significantly reduced, for portfolio managers that seek to find just the optimal outer boundaries and rebalance to a predefined optimal allocation. In that case, the system is comprised by two nonlinear equations. The predefined rebalancing point may be the optimal allocation as derived under the frictionless markets hypothesis for any optimization objective that suits best the portfolio manager. It’s straightforward to observe a further reduction in complexity for investors that just aim to find an optimal symmetric control band around the predefined optimal allocation (a single nonlinear equation). In the fourth section we illustrate how the optimal rebalancing policies, characterized by four parameters $L < l < u < U$, can be derived by solving a certain quasi-variational inequality (QVI).

### 2. Problem formulation and the risky fraction process

We consider the simple two-asset market model, in which the set of securities consists of one bond, whose price $S^0(t)$ is described by the following ordinary differential equation:

$$
\frac{dS^0(t)}{dt} = rS^0(t)dt, \quad S^0(0) = s^0,
$$

and one risky asset with price $S^1(t)$ that is governed by the stochastic differential equation:

$$
\frac{dS^1(t)}{dt} = S^1(t)(\mu dt + \sigma dW_t), \quad S^1(0) = s^1
$$

where $W_t$ is a standard Wiener process defined on a filtered probability space $(\Omega, F, P, F_t)$. We assume that $F_t$ satisfies the usual conditions, namely it is right continuous and $F_0$
includes all $P$-null sets in $F$, and that $\sigma^2 > 0$. Let $\left(\pi^0(t), \pi^1(t)\right)$ be the shareholding process, to be chosen by the portfolio manager, each component of which represents the number of shares for the $i$-th asset at time $t$. It is required to be a piecewise constant, adapted process.

Denote by $V(t) := \sum_{i=0}^{1} \pi^i(t)S^i(t)$ the wealth process or value process, which is strictly positive for all $t \geq 0$. Now we may define the risky fraction process $b^i(t)$ by setting

$$b^i(t) = \frac{\pi^i(t)S^i(t)}{V(t)}, \quad i = 0, 1$$

(2.3)

and for later use we set $b(t) = b^1(t)$. Under the condition of self-financing $V(t)$ satisfies

$$dV(t) = V(t)\left(\sum_{i=0}^{1} b^i(t) \frac{dS^i(t)}{S^i(t)}\right) = b^0(t)r dt + b^1(t)(\mu dt + \sigma dW_t), \quad V(0) = v$$

(2.4)

and we have

$$\frac{dV(t)}{V(t)} = (r + b(t)(\mu - r)) dt + b(t)\sigma dW_t, \quad V(0) = v.$$ 

(2.5)

The risky fraction process was first studied by Morton and Pliska [25]. Using Ito’s formula, they showed that, for the two-asset case, it evolves according to the following stochastic differential equation

$$db_t = b_t(1 - b_t)(\mu - r - \sigma^2 b_t) dt + b_t(1 - b_t)\sigma dW_t.$$ 

(2.6)

To ease calculations in later sections, we adopt the 1-1 transformation proposed recently by Nagai [26], defined by

$$y = \psi(b) := \log b - \log(1 - b)$$

(2.7)

and one may easily observe the form of the inverse mapping $\phi$:

$$h = \phi(y) := \frac{\exp y}{1 + \exp y}.$$ 

(2.8)

Using once again Ito’s formula, the evolution of $y$ is formulated as a geometric Brownian motion with constant drift

$$dy_t = \kappa dt + \sigma dW_t,$$ 

(2.9)

where $\kappa = \mu - r - \frac{\sigma^2}{2}$; a certainly more manageable form compared to (2.6).
We now turn to the specification of the transaction cost, which is essentially the same as in Suzuki and Pliska [31]. If the transformed risky proportion is \( y \) and a transaction is made resulting the new risky proportion \( \tilde{y} \), then the transaction cost incurred at that time is

\[
c(y, \tilde{y}) := K + k |y - \tilde{y}|
\]

(2.10)

where \( K \) and \( k \) are two suitably chosen (so that the scale transformation is accounted for), strictly positive scalars. Thus, the linear component is proportional to the change in transformed proportions and not, as is common in much of the transaction cost literature, proportional to the dollar amount of the transaction. Because of the fixed cost component, it suffices to consider trading strategies of the form \( \{(\tau_n, y_n)\} \), where \( \tau_n \) is the time of the \( n \)th transaction and \( y_n \) the risky proportion that results from the \( n \)th transaction. \( \{(\tau_n, y_n)\} \) must satisfy some standard technical requirements: \( \tau_n \) is a stopping time, \( \tau_n < \tau_{n+1} \), \( \tau_n \to \infty \) as \( n \to \infty \), and \( y_n \) is \( F_{\tau_n} \)-measurable.

We study two different methods for the derivation of the optimal rebalancing strategies and in the application we compare the control bands they suggest. In the first case, that is to be treated in the next section, the portfolio manager’s objective is to minimize the expected cost over a transaction cycle. The expected cost is the sum of the expected transaction cost and the expected opportunity cost (or tracking error). To illustrate our methodology we use a
quadratic function for the deviation of the risky fraction process from the target level in the transformed (by 2.7) scale. The reader should note that a portfolio manager may choose from a large variety of loss functions to represent his/her opportunity costs – the quadratic functions here are chosen for computational simplicity in the application. For example, tracking error may be represented by an exponential function of the deviation of the observed from the desired proportion or a function of HARA type with a suitably chosen risk aversion parameter. The target levels of the risky fraction process in the original scale may for instance be

\[-\pi_1 = \frac{\mu - r}{\sigma^2},\]

the risky asset proportion that maximizes log utility and the portfolio’s exponential growth rate.

\[-\pi_2 = \frac{\mu - r}{\gamma \sigma^2}\]

the risky asset proportion that maximizes HARA utility with risk aversion parameter \(\gamma\).

In the impulse control method, which is treated in the fourth section, the objective is to minimize the expected discounted squared tracking error plus transaction costs over an infinite planning horizon. Let \(\pi\) denote the target proportion of wealth in the risky asset in the transformed scale. Then under an admissible trading strategy \(\{(r_n, y_n)\}\) and given an initial proportion vector \(b(0) = b_0\), the objective function is given by

\[J(y_0, \{(r_n, y_n)\}) := \mathbb{E}\left[\int_0^\infty e^{-\beta t} \left(e^{y(t)} - 1\right)^2 dt + \sum_{n=1}^\infty e^{-\beta \tau_n} c(y(r_{n-1}), y_n)\right].\]  

(2.11)

In (2.11) the first term measures discounted tracking error/opportunity costs and the second discounted transaction costs; \(\beta\) is a discount factor and \(\lambda\) is a constant chosen by the portfolio manager to reflect his/her loss preferences. The portfolio manager seeks an admissible trading strategy minimizing \(J(y_0, \{(r_n, b_n)\})\). Hence, one would like to compute the value function

\[J(y_0) := \inf_{\{(r_n, y_n)\}} J(y_0, \{(r_n, y_n)\})\]  

(2.12)

where the infimum is taken, over all admissible trading strategies, and find the trading strategy that attains this infimum.
3. Minimization of expected cost per transaction cycle

The usual practice for dealing with tracking problems in the presence of constant and proportional transaction costs where a finite number of actions per finite time intervals is required, is to minimize lifetime discounted tracking error plus discounted transaction costs via solving a system of quasi-variational inequalities. This approach has been adopted in Cadenillas and Zapatero [13] for optimal control of an exchange rate and in Suzuki and Pliska [31] for index tracking. In this section, we minimize average tracking error plus transaction cost per transaction cycle. This approach uses simple mathematical tools from diffusion theory; thus, no knowledge of impulse control theory is required. Similar to the qvi approach, estimation of the inner and outer control bands pertains to the solution of a system of nonlinear equations. Unfortunately, these nonlinear equations turn out to be significantly more complex compared to the ones derived from the qvi approach. Nevertheless, one may relatively easily derive them using any software that performs symbolic calculations\(^1\) and solve the resultant system using standard routines that perform algorithms like the Newton-Raphson and its descendants.

Let \( y_t \) be the transformed (by 2.7) proportion of wealth an investor has in the risky asset at time \( t \). In the absence of intervention, \( y_t \) behaves as the geometric Brownian motion (2.9). Deviations from the (pre-specified) optimal fraction \( \pi \), involve an opportunity cost since part of wealth is not optimally invested. We therefore suppose that holding stocks at level \( y_t \) for the transformed risky fraction process incurs opportunity costs at a quadratic rate in the original scale

\[
g(y_t) = \lambda(e^{(y_t-\pi)} - 1)^2.
\]  

(3.1)

Now consider the following control band policy for the transformed risky fraction process: “If the transformed risky fraction process reaches level \( U \) above the target level \( \pi \), reduce its level to \( u \). This transaction incurs a cost of \( K + k(U - u) \). If the transformed risky fraction process reaches level \( L \) below the target level \( \pi \), increase its level to \( l \). This transaction incurs a cost of \( K + k(l - L) \).” Define a cycle to be from one intervention returning the level to \( l \) or

\(^1\) For the calculations of this article we used MATLAB’s Symbolic Math Toolbox.
u from L or U, to the next such intervention; the long-run cost per unit time will be the
expected cost per cycle divided by the expected cycle time, or
\[
\frac{C + A}{B}
\]  \hspace{1cm} (3.2)

where C represents the expected transaction cost per cycle, A denotes the expected
opportunity cost per cycle, and B stands for the expected cycle time.

To derive A, B and C we use standard diffusion theory as in Carlin and Taylor [14], or
Borodin and Salminen [9]. Assume U and L be fixed subject to \( -\infty < L < U < \infty \), and define
\( T(s) = T_s \) be the hitting time of \( s \) for the \( y \) process. Throughout the paper we let
\[
T^* = \min\{T(U), T(L)\} = T(U) \land T(L)
\]  \hspace{1cm} (3.3)

be the first time the process reaches U or L. To proceed, we need to be able to calculate the
following quantities for the transformed risky fraction process:
\[
v_1 (y) = \Pr\{T(U) < T(L)\} \biggm| Y(0) = y \}
\]  \hspace{1cm} (L < y < U)

the probability the process reaches U before L,
\[
v_2 (y) = E\left[ T^* \biggm| Y(0) = y \right]
\]  \hspace{1cm} (L < y < U)

the mean time to reach U or L, and
\[
v_3 (y) = E\left[ \int_0^{T^*} g(Y(t)) \biggm| Y(0) = y \right]
\]  \hspace{1cm} (L < y < U)

for a bounded and continuous function \( g \).

By standard diffusion theory, the aforementioned functions of the \( y_i \) process satisfy the
following differential equations
\[
\kappa \frac{d v_1 }{d y} + \frac{\sigma^2}{2} \frac{d^2 v_1 }{d y^2} = 0 \text{ for } L < y < U, \quad v_1 (L) = 0, \quad v_1 (U) = 1; \]  \hspace{1cm} (3.7)

\[
\kappa \frac{d v_2 }{d y} + \frac{\sigma^2}{2} \frac{d^2 v_2 }{d y^2} = -1 \text{ for } L < y < U, \quad v_2 (L) = v_2 (U) = 0; \]  \hspace{1cm} (3.8)

\[
\kappa \frac{d v_3 }{d y} + \frac{\sigma^2}{2} \frac{d^2 v_3 }{d y^2} = -g(y) \text{ for } L < y < U, \quad v_3 (L) = v_3 (U) = 0. \]  \hspace{1cm} (3.9)

For the solutions of these problems, let
\( s(y) = \exp\left\{ \int^y \left[ 2\kappa / \sigma^2 \right] d\xi \right\}, \) and
\[
S(y) = \int^y s(\eta) d\eta
\] (3.10)

\[
denote the scale function of the \( y_t \) process, and
\[
m(y) = \frac{1}{\sigma^2 s(y)},
\] (3.11)

\[
the speed density of the \( y_t \) process. The solution to (3.4) is
\[
v_1(y) = \frac{S(y) - S(L)}{S(U) - S(L)} \text{ for } L \leq y \leq U.
\] (3.13)

(3.5) is a special case of (3.6) with \( g \) equal to the indicator function. The solutions to these problems are formulated as follows
\[
v_2(y) = 2v_1(y) \int^y \left[ S(U) - S(\xi) \right] m(\xi) d\xi + \left[ 1 - v_1(y) \right] \int^\xi \left[ S(\xi) - S(L) \right] m(\xi) d\xi
\] (3.14)

\[
v_3(y) = 2v_1(y) \int^y \left[ S(U) - S(\xi) \right] m(\xi) g(\xi) d\xi + \left[ 1 - v_1(y) \right] \int^\xi \left[ S(\xi) - S(L) \right] m(\xi) g(\xi) d\xi.
\] (3.15)

Now the scale function for the geometric Brownian motion (2.9) that corresponds to the transformed risky fraction process, is
\[
S(y) = \exp\left( -2\kappa y / \sigma^2 \right)
\] (3.16)

and the speed measure is
\[
m(y) = \exp\left( 2\kappa y / \sigma^2 \right) / \sigma^2
\] (3.17)

Using (3.17) the expected transaction cost per cycle is
\[
C = K + \Pr\{ T(U) > T(L) \} Y(0) = u k(U - u) + \Pr\{ T(U) > T(L) \} Y(0) = l k(U - l)
\]
\[
+ \Pr\{ T(U) > T(L) \} Y(0) = u k(l - U) + \Pr\{ T(U) > T(L) \} Y(0) = l k(l - L)
\] (3.18)
\[
= K + \frac{S(u) + S(l) - 2S(L)}{S(U) - S(L)} k(U - u) + \frac{S(u) + S(l) - 2S(U)}{S(U) - S(L)} k(l - L)
\]

For the expected cycle time, using (3.14), (3.16), (3.17), we obtain
\[
B = (v_1(u) + v_1(l)) v_2(u) + (2 - v_1(u) - v_1(l)) v_2(l)
\] (3.19)
where
Similarly, the expected opportunity cost/tracking error per transaction cycle is

\[ A = (v_1(u) + v_1(l))v_3(u) + (2 - v_1(u) - v_1(l))v_3(l) \]  

(3.21)

where \( v_3(y) = -v_1(y)h(U) - (1 - v_1(y))h(L) \)

(3.22)

and

\[ h(y) = \frac{\lambda \left( 4\kappa^3 + 6\sigma^2 \kappa^2 + 2\sigma^4 \kappa \right) \left( y - \pi \right) + 7\sigma^2 \kappa^2 - \sigma^6 + \left( 2\kappa^4 + \sigma^2 \kappa^2 \right) e^{(2\left(y - \pi\right))}}{\sigma^2 \kappa \left( \sigma^2 + 2\kappa \right) \left( \sigma^2 + \kappa \right)} \]  

(3.23)

\[ - \frac{\lambda \left[ 8\kappa^4 + \sigma^2 \kappa^2 \right] e^{(y - \pi)} + 6\kappa^3 + \sigma^6 S(y - \pi)}{\sigma^2 \kappa \left( \sigma^2 + 2\kappa \right) \left( \sigma^2 + \kappa \right)} \]

Hence, by using 3.18-3.23 we derived the quantities \( \frac{C + A}{B} \). To minimize with respect to \( L, l, u \) and \( U \), one should take the corresponding derivatives and equate them to zero. Since these expressions are lengthy, we omit them for space economy. The derivatives form a system of nonlinear equations, which can be solved computationally using the Newton-Raphson algorithm or one of its successors; numerical results are presented at the sixth section.

Some remarks are worth of considering. First, instead of seeking two optimal rebalancing points (an inner band), one may simplify the problem by considering a single rebalancing point where the process is driven when it reaches \( L \) or \( U \). In this case, he/she would obtain a system of three nonlinear equations, which are significantly simpler than the case displayed before. Moreover, a manager may be satisfied by just rebalancing to his pre-specified optimal choice \( \pi \) (which as mentioned before may be an optimal allocation as derived under the frictionless markets hypothesis and any optimization objective). In that case, the system contains just two nonlinear equations. The expression \( \frac{C + A}{B} \) is derived as follows:
\[ C = K + \Pr[T(U) > T(L) \mid Y(0) = \pi]\pi k(U - \pi) + \Pr[T(L) > T(U) \mid Y(0) = \pi]\pi k(\pi - L), \quad (3.24) \]

\[ = K + v_1(\pi)k(U - \pi) + (1 - v_1(\pi))k(\pi - L) \]

\[ B = v_2(\pi) = 2\frac{(U(S(\pi) - S(L)) + \pi(S(L) - S(U)) + L(S(U) - S(\pi)))}{(S(L) - S(U))\sigma^2} \]

\[ = \frac{2\pi}{\sigma^2} - 2\frac{U}{\sigma^2}\Pr[T(U) > T(L) \mid Y(0) = \pi] - \frac{2L}{\sigma^2}\Pr[T(L) > T(U) \mid Y'(0) = \pi] \]

and

\[ A = v_3(\pi) = -v_1(\pi)h(U) - (1 - v_1(\pi))h(L) \quad (3.26) \]

with \( h(y) \) as in (3.23). Naturally, computations can be reduced even more if one just seeks a symmetric control band around his/her pre-specified rebalancing point (a single nonlinear equation).

The reader should also note that computations are significantly simplified by considering the transformed risky fraction process. For example for the scale function of the original process one would have

\[ s(y) = \exp\left\{ - \int [2\mu(\xi)/\sigma(\xi)^2]d\xi \right\} = \exp\left\{ - \int [2(\mu - r - \sigma^2\xi)/(\sigma^2\xi(1 - \xi))]d\xi \right\} = \]

\[ = 2 \frac{(\log(y) - \log(y - 1))(\mu - r)}{\sigma^2} - \log(y - 1) \]

and

\[ S(y) = y^{-2p}(y - 1)^{-2(p+1)} \quad (3.28) \]

with

\[ p = \frac{\mu - r}{\sigma^2}. \quad (3.29) \]
4. Solution via a quasi-variational inequality

In this section, we show how to solve the portfolio manager’s tracking problem (2.11) by solving a quasi-variational inequality. Let $\phi(.)$ denote a real-valued function on $\mathbb{R}$. Define the minimum cost switching operator $M$, associated with any such function $\phi(.)$ and the transaction cost function $c(.,.)$ by taking

$$\mathbb{M}\phi(y) := \sup_{\tilde{y}} \{ \phi(\tilde{y}) - c(y, \tilde{y}) \}. \quad (4.1)$$

Recall the stochastic differential equation (2.9) satisfied by the transformed risky fraction process and define the second order partial differential operator $L$ by taking

$$L\phi(y) := \frac{1}{2} \sigma^2 \phi''(y) + \kappa \phi'(y). \quad (4.2)$$

With $f(y) := \lambda (\exp(y - \pi) - 1)^2$ denoting the tracking error rate for this problem, by standard methods for impulse control problems (e.g. see Bensoussan [5], Bensoussan and Lions [6], Korn [20]) we are led to the following quasi-variational inequality:

$$\max \{ L\phi(y) - \beta \phi(y) - f(y), M\phi(y) - \phi(y) \} = 0. \quad (4.3)$$

Indeed, if $\phi$ is a twice continuously differentiable function satisfying this qvi as well as some technical growth conditions, then

$$\phi(y) \geq J(y, \{ \tau_n, y^n \}) \quad (4.4)$$

for all $y \in \mathbb{R}$ and all admissible strategies $\{ \tau_n, y^n \}$. If, moreover, the strategy corresponding to $\phi$ is admissible, then it is an optimal strategy and $\phi(.)$ is identical to the value function $J(.)$. The proof of this ‘verification theorem’ is lengthy, technical, and reasonably standard (e.g. see Korn [19] or Bielecki and Pliska [7]), so it will be omitted. The construction of the strategy corresponding to a solution $\phi$ goes as follows. With $\tau_0 = 0$ and $Y(0-) = y_0$ one has

$$\tau_n := \inf \{ t \geq \tau_{n-1} : \phi(y(t-)) = M\phi(y(t-)) \} \quad (4.5)$$

and

$$y^n = \arg \max_{y \in \mathbb{R}} \{ \phi(\tilde{y}) - c(y(\tau_n-), \tilde{y}) \}. \quad (4.6)$$

Note that $\phi$ defines a continuation region

$$C := \{ y \in \mathbb{R} : M\phi(y) < \phi(y) \}, \quad (4.7)$$
as no transactions occur as long as \( y(t) \in C \). But if \( y(t) \in \partial C \) (e.g., if \( y(t) \) hits the boundary of \( C \)), then a transaction immediately occurs, shifting the risky fraction process according to (4.6).

The supremum operator \( M \), for our problem is
\[
M\phi(y) = \sup_{y \in \mathbb{R}} \left\{ \phi(y) - K - k|y - \bar{y}| \right\}
\] (4.8)
thus qvi (4.3) becomes
\[
0 = \max \left\{ \frac{\sigma^2}{2} \phi''(y) + \phi'(y) - \beta \phi(y) - \lambda (\exp(y - \pi) - 1)^2, \sup_{y \in \mathbb{R}} \left\{ \phi(y) - \phi(y) - K - k|y - \bar{y}| \right\} \right\}
\] (4.9)
We now explain how this qvi can be solved. The ordinary differential equation corresponding to (4.9) has a general solution of the form
\[
\phi(y) = C_1 e^{-\tau y} + C_2 e^{-\kappa y} + \tilde{h}(y)
\] (4.10)
where \( \tilde{h} \) is the particular solution of the differential equation given by
\[
\tilde{h}(y) = \frac{A_1 + A_2 \exp(y - \pi) + A_3 \exp(2(y - \pi))}{A_4}
\]
with
\[
A_1 = 6\kappa (\sigma^2 - \beta - 5\sigma^2 \beta + 2(\sigma^2 + \beta^2)) + 4\kappa^2
\]
\[
A_2 = 4\beta(2\sigma^2 + 2\kappa - \beta)
\]
\[
A_3 = -\beta(2\kappa + \sigma^2 - 2\beta)
\]
\[
A_4 = (2\sigma^2 - \beta + 2\kappa)\beta(2\beta - \sigma^2 - 2\kappa)
\]
Here \( C_1 \) and \( C_2 \) are constants depending on boundary conditions and \( x_1, x_2 \) are formulated as follows
\[
x_{1,2} = \frac{\kappa \pm \sqrt{\kappa^2 + 2\sigma^2 \beta}}{\sigma^2}.
\] (4.12)

For most values of the data parameters, it can be shown that there exist four parameters satisfying \( L < l < u < U \) such that the solution of the qvi (4.9) will be of the form
\[
\phi(y) = \begin{cases} 
ky + \left\{ \phi(y) - kl - K \right\} & y \in (-\infty, L] \\
\phi(y) & y \in (L, U) \\
-ky + \left\{ \phi(y) + ku - K \right\} & y \in [U, \infty)
\end{cases}
\] (4.13)
Here (L, U) is the continuation region. For \( y \in (-\infty, L] \) one should immediately rebalance to \( y = l \), and for \( y \in [U, \infty) \) one should immediately rebalance to \( y = u \). It remains to determine the values of the six parameters \( C_1, C_2, L, l, u \) and \( U \). On that purpose, one should solve a system of six nonlinear equations. To derive these equations we note that the function \( \phi(.) \) must be continuous at \( y = l \), so

\[
\phi(L) = kL + \phi(l) - kl - K. 
\] (4.14)

Similarly, we get a second equation

\[
\phi(U) = -kU + \phi(u) + ku - K. 
\] (4.15)

The derivatives at \( y = L \) and \( U \) must be continuous, so

\[
\phi'(L) = k 
\] (4.16)

and

\[
\phi'(U) = -k. 
\] (4.17)

Since \( \bar{y} = l \) minimizes \( \phi(\bar{y}) - K - k (\bar{y} - L) \) the first order necessary condition gives

\[
\phi'(l) = k 
\] (4.18)

and similarly the final equation is

\[
\phi'(u) = -k. 
\] (4.19)

The system of six equations can readily be solved by MATLAB for the six parameters; a detailed numerical illustration is planned in a forthcoming version of the article.
References